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Electromagnetic angular momentum of the electron: One-loop studies

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Abstract

We study angular momentum of the electron stored in its electric and magnetic fields. We use for this purpose quantum electrodynamics in the covariant gauge. We show that a finite one-loop result for such angular momentum can be obtained without invoking any renormalization procedure. We compare it to the classical estimation relying on a short-distance cutoff.

1. Introduction

When electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{B}$) fields cross, the Poynting vector, $\boldsymbol{E} \times \boldsymbol{B}$, tells us that there is a flow of electromagnetic energy. Angular momentum associated with it reads [1]

$$\boldsymbol{J}_{\text{field}} = \int d^3r\, \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}) \tag{1}$$

and we will call it field angular momentum for brevity.

Such a form of angular momentum is quite intriguing if we notice that it is generically non-zero in static electromagnetic fields, where no dynamics seems to be happening at first glance. For example, a charge and a magnet placed at fixed-in-time positions create all around a "circular" flow of the electromagnetic energy producing non-zero angular momentum density. As $\boldsymbol{J}_{\text{field}}$ is a part of total angular momentum, its changes in systems, where total angular momentum is

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conserved, induce changes in angular momentum associated with other degrees of freedom (e.g. a much more intuitive mechanical angular momentum). A famous example of this phenomenon is known as the Feynman's disk paradox, where one considers an electrically charged disk and a superconducting wire carrying an electric current (see Secs. 17-4 and 27-6 of [2] and [3–5]). When temperature rises, the current disappears and the disc starts rotating. This seemingly violates angular momentum conservation if one forgets about conversion of vanishing field angular momentum into mechanical angular momentum of the disc. It is thus reasonable to argue that $\boldsymbol{J}_{\text{field}}$ is a fundamentally-important counterintuitive quantity deserving in-depth theoretical and experimental studies.

One of the simplest settings for its discussion is found by considering a physical object at rest having the charge q and the magnetic moment $\boldsymbol{\mu}$. Far away from it, where not only details of its structure but also quantum effects can be neglected, its electric and magnetic fields are well-approximated by classical expressions [1]

$$\boldsymbol{E} = \frac{q\boldsymbol{r}}{4\pi r^3}, \quad \boldsymbol{B} = \frac{3(\boldsymbol{\mu}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{\mu}}{4\pi r^3}, \tag{2}$$

where $r = |\boldsymbol{r}|$ and $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. The position vector $\boldsymbol{r}$ goes from the object to the point, where the fields are discussed.

A quick look at density of such angular momentum, which we depict in Fig. 1, shows that one can anticipate a non-trivial result for field angular momentum. To quantify this expectation, one restricts the integration in (1) to $r \geq r_c$, where r_c is large-enough to ensure that the use of (2) is justified. It is then a simple exercise to show that [6]

$$\boldsymbol{J}_{\text{field}} = \boldsymbol{\mu}\frac{q}{6\pi r_c}. \tag{3}$$

Two remarks are in order now.

First, $\boldsymbol{J}_{\text{field}}$ is parallel (anti-parallel) to the magnetic moment for positively (negatively) charged objects. The same relation between spin angular momentum and the magnetic moment is observed for protons and electrons.

Second, it is of interest to find what result could be obtained if one employs some classically-motivated value for the cutoff r_c [6]. A characteristic length-scale that can be used for such a purpose exists within the century-old classical theory of the electron (see e.g. [7]). It is known as the classical electron radius

$$r_0 = \frac{e^2}{4\pi m}, \tag{4}$$

where m is the electron's mass and $e < 0$ is its charge. Leaving aside for the moment the question of whether it is justified to use (2) at such a short distance from the charge, one may take the "classical" electron as our physical object, set $q = e$, and assume that $r_c = O(r_0)$. All this results in

$$\boldsymbol{J}_{\text{field}} = O\left(\boldsymbol{\mu}\frac{m}{e}\right) \tag{5}$$

suggesting that field angular momentum of the "classical" electron is of the order of electron's spin if one additionally takes into account that $|\boldsymbol{\mu}|$ is of the order of the Bohr magneton

$$\frac{|e|}{2m} \tag{6}$$

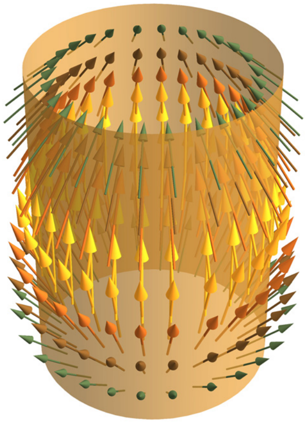

Fig. 1. Density of angular momentum of electromagnetic fields (2). For the clarity of presentation, we show $r^4 \mathfrak{J}_{\text{field}}$ on the surface $x^2 + y^2 = \text{const}$, where $\boldsymbol{r} = (x, y, z)$, the $z = 0$ plane cuts the cylinder in half, and $\mathfrak{J}_{\text{field}} = \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B})$. The magnetic moment $\boldsymbol{\mu}$ is anti-aligned with the z-axis, it points downward the cylinder. The charge $q < 0$.

for the electron. While such an estimation clearly cannot be treated too seriously, it is interesting to set it against the outcome of a fully quantum calculation.

The purpose of this work is to compute field angular momentum of the electron in the framework of quantum electrodynamics (QED). Such a calculation not only comprehensively accounts for the quantum effects, but it also does not rely on a short-distance cutoff. It is therefore interesting to inquire, and in fact a priori unknown, whether the result of such a calculation will be finite. We find it thus remarkable that a finite non-trivial result for such a physically interesting quantity can be obtained. It comes from our one-loop calculation, which does not involve any renormalization procedure. There are different ways how one can place this result in a wider context.

On the one hand, it provides one more physical quantity characterizing the electron, arguably one of the most important subatomic particles. In some sense, such a result is similar to the Schwinger's prediction for the electron's anomalous magnetic moment, which also comes from a one-loop calculation and provides a basic insight into the properties of the electron.

On the other hand, our work can be seen as a part of a larger program targeting characterization of all components of angular momentum of the electron. So far there have been only a few attempts in this direction [8–11], and none of them studied field angular momentum that we discuss here. A similar program is being carried out for nucleons, where various calculations have been set against experimental data (see e.g. [12,13] for recent review articles).

The outline of this paper is the following. We briefly explain in Sec. 2 how our calculations will be carried out. The actual computations are presented in Sec. 3, where we study field angular momentum of the electron with the help of three-dimensional (3D) cutoff and Pauli-Villars regularizations. Our results are then discussed in Sec. 4. The paper ends with Appendix A, where our conventions are briefly summarized.

2. Basic equations

We start with the QED Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}\left(\mathrm{i}\gamma^\mu \partial_\mu - m_{\mathrm{o}}\right)\psi - e_{\mathrm{o}} \overline{\psi} \gamma^\mu \psi A_\mu, \tag{7}$$

where m_o and $e_o < 0$ are the bare mass and charge of the electron, respectively, and the electromagnetic and fermionic fields are defined as always [14]. The theory is canonically quantized in the standard way after adding the gauge-fixing term

$$-\frac{1}{2}\left(\partial_\mu A^\mu\right)^2 \tag{8}$$

to the Lagrangian density [14]. Such a choice leads to Feynman-gauge QED (we will argue below that our results are the same in any covariant gauge). We also mention that all fields from now on will be either Heisenberg- or interaction-picture operators. The latter will be distinguished from the former by the index I.

Next, we replace the classical fields in (1) with operators, impose normal ordering, which we denote by ::, and rewrite the resulting expression to get

$$J^i_{\text{field}} = \int d^3z\, \varepsilon^{imn} z^m :F_{0j} F_{jn}: . \tag{9}$$

We call (9) the field angular momentum operator. It appears in the so-called Belinfante [15] and Ji [16] decompositions of total angular momentum, where it has the physical interpretation of the photon total angular momentum operator [12]–see this reference also for a comprehensive discussion of different total angular momentum decompositions in QED. This operator is gauge invariant and so its expectation value should be measurable in principle. Such an important property should not be taken for granted. Indeed, in the so-called Jaffe-Manohar [17] decomposition of total angular momentum, the following operator accounts for angular momentum of the electromagnetic field

$$\int d^3z\, \varepsilon^{imn} :F_{m0} A_n: + \int d^3z\, \varepsilon^{imn} z^m :F_{j0} \partial_n A_j:, \tag{10}$$

where the first (second) term has the physical interpretation of the photon spin (orbital) angular momentum operator [12]. Operator (10) is gauge non-invariant in the presence of charges, which can be easily checked. Differences between (9) and (10) nicely illustrate the fact that in the interacting theory such as QED, where electromagnetic and fermionic fields are coupled, there is no unique division of total angular momentum into electromagnetic and fermionic components. Still, the study of (9) is well-motivated physically and it provides a finite gauge invariant result relevant for understanding of angular momentum of the electron within the Belinfante and Ji decompositions.

We will compute expectation value of (9) with the help of the bare perturbation theory in the QED ground state $|\mathbf{\Omega} s\rangle$ with one net electron (the difference between the number of electrons and positrons in such a state is $+1$). For this purpose, we will use imaginary time evolution starting from the ground state of the non-interacting theory with one electron at rest having spin projection onto the z-axis

$$s_z = \pm\frac{1}{2}. \tag{11}$$

We refer to such a state as $|\mathbf{0}s\rangle$ and mention that the expectation value of the total angular momentum operator in states $|\mathbf{0}s\rangle$ and $|\mathbf{\Omega}s\rangle$ equals $s_z\delta^{i3}$.

Adopting the results of [18] to our problem, we get

$$\langle \boldsymbol{J}_{\text{field}}\rangle_{\mathbf{\Omega}s} = \lim_{T\to\infty(1-\mathrm{i}0)} \frac{\langle \mathbf{0}s|\mathbb{T}\boldsymbol{J}^I_{\text{field}} \exp(-\mathrm{i}\int_T d^4x\, \mathcal{H}^I_{\text{int}})|\mathbf{0}s\rangle}{\langle \mathbf{0}s|\mathbb{T}\exp(-\mathrm{i}\int_T d^4x\, \mathcal{H}^I_{\text{int}})|\mathbf{0}s\rangle}, \tag{12}$$

where

$$\langle\cdots\rangle_{\boldsymbol{\Omega}s} = \frac{\langle\boldsymbol{\Omega}s|\cdots|\boldsymbol{\Omega}s\rangle}{\langle\boldsymbol{\Omega}s|\boldsymbol{\Omega}s\rangle}, \tag{13}$$

$$\mathcal{H}^I_{\mathrm{int}}(x) = e_{\mathrm{o}} :\overline{\psi}_I(x)\gamma^\mu\psi_I(x): A^I_\mu(x), \tag{14}$$

$$\int_T d^4x = \int_{-T}^{T} dx^0 \int d^3x\,, \tag{15}$$

and $\mathbb{T}$ is the time-ordering operator.

Equation (12) can be substantially simplified. Indeed, with the help of our results presented in [11], it can be rigorously shown that one can safely do the replacement

$$\lim_{T\to\infty(1-\mathrm{i}0)} \int_T d^4x \to \int d^4x \tag{16}$$

if the calculations of field angular momentum of the electron are infrared-regularized. This leads to

$$\langle \boldsymbol{J}_{\mathrm{field}}\rangle_{\boldsymbol{\Omega}s} = \frac{\langle\mathbf{0}s|\mathbb{T}\boldsymbol{J}^I_{\mathrm{field}}\exp(-\mathrm{i}\int d^4x\,\mathcal{H}^I_{\mathrm{int}})|\mathbf{0}s\rangle}{\langle\mathbf{0}s|\mathbb{T}\exp(-\mathrm{i}\int d^4x\,\mathcal{H}^I_{\mathrm{int}})|\mathbf{0}s\rangle} \tag{17}$$

in accordance with the standard textbook description of the imaginary time evolution technique [18]. Replacement (16), however, should not be taken for granted, which we illustrate in [11].

Finally, we need the interaction-picture version of $\boldsymbol{J}_{\mathrm{field}}$, which is obtained by replacing the Heisenberg-picture operators with their interaction-picture counterparts. This can be seen by using canonical commutation relations

$$[A_\mu(x^0,\boldsymbol{y}), A_\nu(x)] = 0 \tag{18}$$

to show that the last term of

$$\big(\partial_\mu A_\nu(x)\big)_I = \partial_\mu A^I_\nu(x) + \mathrm{i}\eta_{\mu0}\int d^3y\,\Big([\mathcal{H}_{\mathrm{int}}(x^0,\boldsymbol{y}), A_\nu(x)]\Big)_I \tag{19}$$

vanishes (see e.g. [14] for the transformation relating the two pictures).

3. Field angular momentum

We will compute here the expectation value of the field angular momentum operator using two regularization methods. Most of the computations in this section, however, can be done without referring to any regularization technique. Such results will be collected in Sec. 3.1. They will be then adapted to calculations based on either the 3D cutoff (Sec. 3.2) or Pauli-Villars (Sec. 3.3) regularization.

3.1. Base formulae

To calculate the expectation value of field angular momentum operator (9), we expand (17) in the series in e_{o}

$$\langle J^i_{\mathrm{field}}\rangle_{\boldsymbol{\Omega}s} = -\frac{1}{2V}\int d^4x\,d^4y\,\langle\mathbf{0}s|\mathbb{T}(\boldsymbol{J}^I_{\mathrm{field}})^i\mathcal{H}^I_{\mathrm{int}}(x)\mathcal{H}^I_{\mathrm{int}}(y)|\mathbf{0}s\rangle + O(e_{\mathrm{o}}^4), \tag{20}$$

where we have replaced the denominator of (17) with

$$V = \langle \mathbf{0}s|\mathbf{0}s\rangle = \int \frac{d^3x}{(2\pi)^3} \tag{21}$$

because we work in the quadratic order in e_{o} and $\langle \mathbf{0}s|\boldsymbol{J}^I_{\mathrm{field}}|\mathbf{0}s\rangle = 0$. Normalization factor (21) of delocalized one-electron states is formally infinite, but it unambiguously cancels down during calculations (see e.g. discussion in [11]). This is a well-known feature of calculations done in the plane-wave basis.

The electromagnetic and fermionic contributions, to the matrix element in (20), factor out

$$\langle \mathbf{0}s|\mathbb{T}(\boldsymbol{J}^I_{\mathrm{field}})^i \mathcal{H}^I_{\mathrm{int}}(x)\mathcal{H}^I_{\mathrm{int}}(y)|\mathbf{0}s\rangle = e_{\mathrm{o}}^2 \mathcal{E}^i_{\mu\nu}(x,y)\mathcal{F}^{\mu\nu}(x,y), \tag{22a}$$

$$\mathcal{E}^i_{\mu\nu}(x,y) = \varepsilon^{imn}\int d^3z\, z^m \langle 0|\mathbb{T}:F^I_{0j}(z)F^I_{jn}(z): A^I_\mu(x)A^I_\nu(y)|0\rangle, \tag{22b}$$

$$\mathcal{F}^{\mu\nu}(x,y) = \langle \mathbf{0}s|\mathbb{T}:\overline{\psi}_I(x)\gamma^\mu\psi_I(x)::\overline{\psi}_I(y)\gamma^\nu\psi_I(y):|\mathbf{0}s\rangle, \tag{22c}$$

where $|0\rangle$ is the vacuum state of the non-interacting theory and z^0 is dropped from the list of arguments of $\mathcal{E}^i_{\mu\nu}$ for the sake of brevity (the same is done for $\mathcal{A}^i_{\mu\nu}$ and $\mathcal{B}^i_{\mu\nu}$ that will be introduced below).

To compute the fermionic matrix element, we apply the Wick's theorem to (22c)

$$\begin{aligned}\mathcal{F}^{\mu\nu}(x,y) &= \langle \mathbf{0}s|\overline{\psi}_I(x)\gamma^\mu\psi_I(x)\overline{\psi}_I(y)\gamma^\nu\psi_I(y)|\mathbf{0}s\rangle \\ &\quad + \frac{1}{2}\langle \mathbf{0}s|\overline{\psi}_I(x)\gamma^\mu\psi_I(x)\overline{\psi}_I(y)\gamma^\nu\psi_I(y)|\mathbf{0}s\rangle \\ &\quad + (x,\mu \leftrightarrow y,\nu \text{ on all terms}).\end{aligned} \tag{23}$$

This can be evaluated with the following contractions

$$\psi_I(x)\overline{\psi}_I(y) = \langle 0|\mathbb{T}\psi_I(x)\overline{\psi}_I(y)|0\rangle = \mathrm{i}\int \frac{d^4p}{(2\pi)^4}\frac{\gamma\cdot p + m_{\mathrm{o}}}{p^2 - m_{\mathrm{o}}^2 + \mathrm{i}0}e^{-\mathrm{i}p\cdot(x-y)}, \tag{24}$$

$$\langle \mathbf{0}s|\overline{\psi}_I(x) = \frac{\overline{u}_s}{(2\pi)^{3/2}}e^{\mathrm{i}f\cdot x}, \quad \frac{u_s}{(2\pi)^{3/2}}e^{-\mathrm{i}f\cdot x} = \psi_I(x)|\mathbf{0}s\rangle, \tag{25}$$

where

$$f = (m_{\mathrm{o}}, \mathbf{0}) \tag{26}$$

and the u_s bispinors, describing an electron at rest with the spin projection onto the z-axis given by (11), are provided in Appendix A.

After a few elementary steps, we arrive at

$$\begin{aligned}\mathcal{F}^{\mu\nu}(x,y) &= \frac{\mathrm{i}}{(2\pi)^3}\int \frac{d^4p}{(2\pi)^4}\frac{\overline{u}_s\gamma^\mu(\gamma\cdot p + m_{\mathrm{o}})\gamma^\nu u_s}{p^2 - m_{\mathrm{o}}^2 + \mathrm{i}0}e^{\mathrm{i}(f-p)\cdot(x-y)} \\ &\quad + \frac{V}{2}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{\mathrm{Tr}\big[(\gamma\cdot p + m_{\mathrm{o}})\gamma^\mu(\gamma\cdot q + m_{\mathrm{o}})\gamma^\nu\big]}{(p^2 - m_{\mathrm{o}}^2 + \mathrm{i}0)(q^2 - m_{\mathrm{o}}^2 + \mathrm{i}0)}e^{\mathrm{i}(p-q)\cdot(x-y)} \\ &\quad + (x,\mu \leftrightarrow y,\nu \text{ on all terms})\end{aligned} \tag{27}$$

and, to avoid any confusion, we mention that throughout this work there is no summation over s in bispinor matrix elements $\overline{u}_s \cdots u_s$ (we do not average over spin polarizations).

To simplify (27), we need the following well-known representation-independent identity

$$\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho + \eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu - \mathrm{i}\varepsilon^{\sigma\mu\nu\rho}\gamma_\sigma\gamma^5, \tag{28}$$

where $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. We also need

$$\mathrm{Tr}\Big[(\gamma\cdot p + m_\mathrm{o})\gamma^\mu(\gamma\cdot q + m_\mathrm{o})\gamma^\nu\Big] = 4\Big[p^\mu q^\nu + p^\nu q^\mu - \eta^{\mu\nu}(p\cdot q - m_\mathrm{o}^2)\Big], \tag{29}$$

$$\overline{u}_s\gamma^\mu u_s = \eta^{\mu 0}, \quad \overline{u}_s\gamma^\mu\gamma^\nu u_s = \eta^{\mu\nu} - 2\mathrm{i}s_z\varepsilon^{0\mu\nu 3}, \quad \varepsilon^{\sigma\mu\nu\rho}\overline{u}_s\gamma_\sigma\gamma^5 u_s = 2s_z\varepsilon^{\mu\nu\rho 3}, \tag{30}$$

which can be easily verified in the standard representation of γ matrices (the same results are obtained in all representations, which are unitarily similar to the standard one: Weil, Majorana, etc.). Having these expressions at hand, we would like to remark that field angular momentum will be s_z-dependent.[1] As a result, we learn from (30) that our calculations will critically depend on the four-dimensional Levi-Civita symbol, which is troublesome in dimensional regularization (see e.g. Appendix B.2 of [19]). In fact, the 3D version of this symbol has already appeared in the field angular momentum operator, whose definition is heavily rooted in dimensionality of the physical space. These complications discourage us from using dimensional regularization in the subsequent sections.

Combining (27) with (28)–(30), the fermionic matrix element can be not only simplified but also decomposed into symmetric and anti-symmetric parts with respect to the $\mu \leftrightarrow \nu$ transformation

$$\mathcal{F}^{\mu\nu}(x,y) = \mathcal{F}^{\mu\nu}_{\mathrm{sym}}(x,y) + \mathcal{F}^{\mu\nu}_{\mathrm{asym}}(x,y), \tag{31a}$$

$$\begin{aligned}\mathcal{F}^{\mu\nu}_{\mathrm{sym}}(x,y) = {} & \frac{\mathrm{i}}{(2\pi)^3}\int\frac{d^4p}{(2\pi)^4}\frac{p^\mu\eta^{\nu 0} + p^\nu\eta^{\mu 0} - p^0\eta^{\mu\nu} + m_\mathrm{o}\eta^{\mu\nu}}{p^2 - m_\mathrm{o}^2 + \mathrm{i}0}e^{\mathrm{i}(f-p)\cdot(x-y)} \\ & + 2V\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{p^\mu q^\nu + p^\nu q^\mu - \eta^{\mu\nu}(p\cdot q - m_\mathrm{o}^2)}{\big(p^2 - m_\mathrm{o}^2 + \mathrm{i}0\big)\big(q^2 - m_\mathrm{o}^2 + \mathrm{i}0\big)}e^{\mathrm{i}(p-q)\cdot(x-y)} \\ & + (x \leftrightarrow y \text{ on all terms}),\end{aligned} \tag{31b}$$

$$\mathcal{F}^{\mu\nu}_{\mathrm{asym}}(x,y) = \frac{2s_z}{(2\pi)^3}\int\frac{d^4p}{(2\pi)^4}\frac{\varepsilon^{0\mu\nu 3}m_\mathrm{o} - \varepsilon^{\sigma\mu\nu 3}p_\sigma}{p^2 - m_\mathrm{o}^2 + \mathrm{i}0}e^{\mathrm{i}(f-p)\cdot(x-y)} - (x \leftrightarrow y). \tag{31c}$$

To compute the electromagnetic matrix element, we again make use of the Wick's theorem

$$\mathcal{E}^i_{\mu\nu}(x,y) = \varepsilon^{imn}\int d^3z\, z^m\, \overbracket{F^I_{0j}(z)A^I_\mu}(x)\,\overbracket{F^I_{jn}(z)A^I_\nu}(y) + (x,\mu \leftrightarrow y,\nu). \tag{32}$$

Then, we need the interaction-picture photon propagator in the Feynman gauge

$$\overbracket{A^I_\mu(x)A^I_\nu}(y) = \langle 0|\mathbb{T}A^I_\mu(x)A^I_\nu(y)|0\rangle = -\mathrm{i}\int\frac{d^4p}{(2\pi)^4}\frac{\eta_{\mu\nu}}{p^2 + \mathrm{i}0}e^{-\mathrm{i}p\cdot(x-y)} \tag{33}$$

and the identity

$$\langle 0|\mathbb{T}\partial_\alpha A^I_\beta(x)A^I_\gamma(y)|0\rangle = \frac{\partial}{\partial x^\alpha}\langle 0|\mathbb{T}A^I_\beta(x)A^I_\gamma(y)|0\rangle, \tag{34}$$

which can be trivially proved with (18).

[1] The same is observed in (5), if we note that the electron's magnetic moment also depends on the spin projection.

Combining these simple results, we get

$$\overbrace{F^I_{\alpha\beta}(z)A^I_\gamma}(x) = \int \frac{d^4p}{(2\pi)^4} \frac{p_\alpha \eta_{\beta\gamma} - p_\beta \eta_{\alpha\gamma}}{p^2 + \mathrm{i}0} e^{-\mathrm{i}p\cdot(x-z)}. \tag{35}$$

Quite interestingly, if we would use the general covariant gauge photon propagator [14,20], which is obtained by replacement

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + \frac{1-\xi}{\xi} \frac{p_\mu p_\nu}{p^2 + \mathrm{i}0} \tag{36}$$

in (33), we would get the same result for (35) for all parameters ξ labeling various covariant gauge choices. This shows that our results are gauge independent within the family of all covariant gauges, which is a welcome feature.

Using (35) to evaluate (32), we obtain

$$\mathcal{E}^i_{\mu\nu}(x,y) = \varepsilon^{imn} \int d^3z\, z^m \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{\Delta_{\mu\nu,n}(p,q) + \Delta_{\nu\mu,n}(q,p)}{(p^2+\mathrm{i}0)(q^2+\mathrm{i}0)} e^{-\mathrm{i}p\cdot x + \mathrm{i}q\cdot y + \mathrm{i}(p-q)\cdot z}, \tag{37a}$$

$$\Delta_{\mu\nu,n}(p,q) = (p_0\eta_{j\mu} - p_j\eta_{0\mu})(q_n\eta_{j\nu} - q_j\eta_{n\nu}). \tag{37b}$$

Next, we use

$$\int \frac{d^3z}{(2\pi)^3} z^m e^{\mathrm{i}z\cdot(\boldsymbol{q}-\boldsymbol{p})} = \frac{\mathrm{i}}{2}\left(\frac{\partial}{\partial p^m} - \frac{\partial}{\partial q^m}\right)\delta(\boldsymbol{p}-\boldsymbol{q}), \tag{38}$$

and integrate by parts. As can be easily checked, such integration by parts does not generate boundary contributions.

Finally, introducing

$$\tilde{q} = (q^0, \boldsymbol{p}), \tag{39}$$

we derive

$$\mathcal{E}^i_{\mu\nu}(x,y) = \mathcal{A}^i_{\mu\nu}(x,y) + \mathcal{B}^i_{\mu\nu}(x,y), \tag{40a}$$

$$\mathcal{A}^i_{\mu\nu}(x,y) = \frac{1}{2}\varepsilon^{imn}(x^m + y^m) \int \frac{d^4p}{(2\pi)^4} \frac{dq^0}{2\pi} e^{-\mathrm{i}p\cdot x + \mathrm{i}\tilde{q}\cdot y + \mathrm{i}(p^0-q^0)z^0} \cdot \frac{\Delta_{\mu\nu,n}(p,\tilde{q}) + \Delta_{\nu\mu,n}(\tilde{q},p)}{(p^2+\mathrm{i}0)(\tilde{q}^2+\mathrm{i}0)}, \tag{40b}$$

$$\mathcal{B}^i_{\mu\nu}(x,y) = -\frac{\mathrm{i}}{2}\varepsilon^{imn} \int \frac{d^4p}{(2\pi)^4} \frac{dq^0}{2\pi} e^{-\mathrm{i}p\cdot x + \mathrm{i}\tilde{q}\cdot y + \mathrm{i}(p^0-q^0)z^0} \cdot \left(\frac{\partial}{\partial p^m} - \frac{\partial}{\partial q^m}\right) \frac{\Delta_{\mu\nu,n}(p,q) + \Delta_{\nu\mu,n}(q,p)}{(p^2+\mathrm{i}0)(q^2+\mathrm{i}0)}\bigg|_{\boldsymbol{q}=\boldsymbol{p}}. \tag{40c}$$

We get by collecting (20), (22), (31), and (40)

$$\langle J^i_{\text{field}} \rangle_{\boldsymbol{\Omega}s} = -\frac{e_0^2}{2V} \int d^4x\, d^4y \left[\mathcal{A}^i_{\mu\nu}(x,y) + \mathcal{B}^i_{\mu\nu}(x,y)\right] \mathcal{F}^{\mu\nu}(x,y) + O(e_0^4), \tag{41}$$

where the $\mathcal{A}^i_{\mu\nu}$ term can be dropped because it leads to the integral of the form

$$\int d^3x\, d^3y\, (\boldsymbol{x}+\boldsymbol{y})^m e^{\mathrm{i}\boldsymbol{P}\cdot(\boldsymbol{x}-\boldsymbol{y})} = 0 \tag{42}$$

with $\boldsymbol{P}$ being some combination of 3-momenta.

In the end, we arrive at the unregularized expression for field angular momentum of the electron

$$\langle J^i_{\text{field}}\rangle_{\boldsymbol{\Omega}s} = -2\mathrm{i}e_\mathrm{o}^2 s_z \int \frac{d^4p}{(2\pi)^4} \frac{\delta^{i3}[2(p^0-m_\mathrm{o})^2+\omega_{\boldsymbol{p}}^2]-p_i p_3}{(p^2-m_\mathrm{o}^2+\mathrm{i}0)[(p-f)^2+\mathrm{i}0]^2} + O(e_\mathrm{o}^4), \tag{43}$$

where $\omega_{\boldsymbol{p}} = |\boldsymbol{p}|$. This expression, unlike (40), is time, i.e., z^0-independent. It is an anticipated feature because that expectation value is computed in an eigenstate of the system and $\boldsymbol{J}_{\text{field}}$ has no explicit time dependence. We also note that field angular momentum of the electron does not have the s_z-independent component. This can be explained from two different viewpoints.

First, such a component can arise only from the symmetric part of fermionic matrix element (31b). During evaluation of $\int d^4x\, d^4y\, \mathcal{B}^i_{\mu\nu}(x,y)\mathcal{F}^{\mu\nu}_{\text{sym}}(x,y)$, however, one encounters contractions between symmetric and anti-symmetric in $\mu \leftrightarrow \nu$ tensors, which make such an integral equal to zero. Second, after averaging over spin projections, field angular momentum of the electron should vanish and so its s_z-independent component cannot exist. It is so because after such an operation, there is no preferred direction in the three-dimensional real space.

Until now, we have gone quite far without using any regularization. To assign a value to expression (43), we need to specify a regularization scheme. We will discuss two options below.

3.2. 3D cutoff regularization

The idea here is to regularize calculations from Sec. 3.1 by cutting off 3-momenta in expressions for propagators. This can be achieved by the following modification of electromagnetic propagator (33)

$$d^4p \to [d^4p] = d^4p\, \theta(\Lambda_c - \omega_{\boldsymbol{p}})\theta(\omega_{\boldsymbol{p}} - \lambda_c), \tag{44}$$

where θ is the Heaviside step function and the infrared (IR) and ultraviolet (UV) cutoffs are denoted as λ_c and Λ_c, respectively. Alternatively, one may implement the UV cutoff in fermionic propagator (24) while keeping the IR one in the electromagnetic propagator (application of the IR cutoff to the fermionic propagator is questionable as our imaginary time evolution starts from the zero-momentum state).

If we redo the calculations from Sec. 3.1 with either of the above-outlined options, we will find that

$$\langle \boldsymbol{J}_{\text{field}}\rangle_{\boldsymbol{\Omega}s} = \lim_{\substack{\Lambda_c\to\infty \\ \lambda_c\to 0}} \langle \boldsymbol{J}_{\text{field}}\rangle^{\lambda_c\Lambda_c}_{\boldsymbol{\Omega}s} + O(e_\mathrm{o}^4), \tag{45a}$$

$$\langle J^i_{\text{field}}\rangle^{\lambda_c\Lambda_c}_{\boldsymbol{\Omega}s} = -2\mathrm{i}e_\mathrm{o}^2 s_z \delta^{i3} \int \frac{[d^4p]}{(2\pi)^4} \frac{2(p^0-m_\mathrm{o})^2+(p_1)^2+(p_2)^2}{(p^2-m_\mathrm{o}^2+\mathrm{i}0)[(p-f)^2+\mathrm{i}0]^2}. \tag{45b}$$

To evaluate (45b), we first integrate over p^0 using the residue theorem and then do the integration in the 3D $\boldsymbol{p}$-space. The order of angular and radial integrations in the $\boldsymbol{p}$-space does not matter since the two operations commute when the radial integration is done on a bounded interval. If that would not be the case, then the radial integration, when performed before the angular one, would produce a meaningless logarithmically divergent result.

Leaving the radial integration for the last step of evaluation of (45b), we find

$$\langle J^i_{\rm field}\rangle^{\lambda_c\Lambda_c}_{\mathbf{\Omega}s} = -\frac{e_{\rm o}^2 s_z \delta^{i3}}{6\pi^2}\int_{\lambda_c}^{\Lambda_c} d\omega_{\boldsymbol{p}} \frac{m_{\rm o}^2}{\varepsilon_{\boldsymbol{p}}(\omega_{\boldsymbol{p}}+\varepsilon_{\boldsymbol{p}})^2}, \tag{46}$$

where $\varepsilon_{\boldsymbol{p}} = \sqrt{m_{\rm o}^2+\omega_{\boldsymbol{p}}^2}$. This can be computed after changing the integration variable to y given by (see e.g. Sec. 2.25 of [21])

$$y = \left(\omega_{\boldsymbol{p}}/m_{\rm o} + \sqrt{1+(\omega_{\boldsymbol{p}}/m_{\rm o})^2}\right)^{-2}, \tag{47}$$

which turns the integral in (46) into

$$\int_{y(\Lambda_c)}^{y(\lambda_c)} \frac{dy}{2}, \tag{48}$$

where $y(\omega_{\boldsymbol{p}})$ is given by the right-hand side of (47). In the end, we get

$$\langle J^i_{\rm field}\rangle_{\mathbf{\Omega}s} = -s_z\delta^{i3}\frac{e_{\rm o}^2}{12\pi^2} + O(e_{\rm o}^4). \tag{49}$$

3.3. Pauli-Villars regularization

We will employ the Pauli-Villars regularization in this section [22]. In its simplest version [23,24], this is systematically done by modifying the Lagrangian density so that it reads

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\mu A^\mu\right)^2 + \frac{\lambda^2}{2}A_\mu A^\mu + \overline{\psi}\left(\mathrm{i}\gamma^\mu\partial_\mu - m_{\rm o}\right)\psi \\ &+\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\tilde{A}^\mu\right)^2 - \frac{\Lambda^2}{2}\tilde{A}_\mu\tilde{A}^\mu + \overline{\tilde{\psi}}\left(\mathrm{i}\gamma^\mu\partial_\mu - \Lambda\right)\tilde{\psi} \\ &- e_{\rm o}(\overline{\psi}\gamma^\mu\psi + \overline{\tilde{\psi}}\gamma^\mu\tilde{\psi})(A_\mu+\tilde{A}_\mu),\end{aligned} \tag{50}$$

where $\tilde{\psi}$ and $\tilde{A}_\mu$ are the Pauli-Villars bosonic ghost fields and the mass term has been added for real photons.

The IR regularization is controlled by $\lambda \ll m_{\rm o}$ entering the electromagnetic propagator, which now reads

$$A^I_\mu(x)A^I_\nu(y) = -\mathrm{i}\int \frac{d^4p}{(2\pi)^4}\frac{\eta_{\mu\nu}}{p^2-\lambda^2+\mathrm{i}0}e^{-\mathrm{i}p\cdot(x-y)}, \tag{51}$$

while the UV regularization is supposed to be controlled by $\Lambda \gg m_{\rm o}$.

To see if the latter really happens, we replace $\mathcal{H}^I_{\rm int}$ in (20) with

$$e_{\rm o}(:\overline{\psi}_I\gamma^\mu\psi_I: + :\overline{\tilde{\psi}}_I\gamma^\mu\tilde{\psi}_I:)(A^I_\mu + \tilde{A}^I_\mu) \tag{52}$$

and redefine $|\mathbf{0}s\rangle$ so that it is the state with one real electron at rest in the spin state s and zero real photons and ghost particles. The resulting expression for field angular momentum of the electron depends on the product of "electromagnetic"

$$\langle 0|\mathbb{T}(\boldsymbol{J}^I_{\text{field}})^i A^I_\mu(x)A^I_\nu(y) + \mathbb{T}(\boldsymbol{J}^I_{\text{field}})^i A^I_\mu(x)\tilde{A}^I_\nu(y) + \mathbb{T}(\boldsymbol{J}^I_{\text{field}})^i \tilde{A}^I_\mu(x)A^I_\nu(y) + \mathbb{T}(\boldsymbol{J}^I_{\text{field}})^i \tilde{A}^I_\mu(x)\tilde{A}^I_\nu(y)|0\rangle \quad (53)$$

and "fermionic"

$$\langle \mathbf{0}s|\mathbb{T}:\overline{\psi}_I(x)\gamma^\mu\psi_I(x)::\overline{\psi}_I(y)\gamma^\nu\psi_I(y): + \mathbb{T}:\overline{\psi}_I(x)\gamma^\mu\psi_I(x)::\overline{\tilde{\psi}}_I(y)\gamma^\nu\tilde{\psi}_I(y): + \mathbb{T}:\overline{\tilde{\psi}}_I(x)\gamma^\mu\tilde{\psi}_I(x)::\overline{\psi}_I(y)\gamma^\nu\psi_I(y): + \mathbb{T}:\overline{\tilde{\psi}}_I(x)\gamma^\mu\tilde{\psi}_I(x)::\overline{\tilde{\psi}}_I(y)\gamma^\nu\tilde{\psi}_I(y):|\mathbf{0}s\rangle \quad (54)$$

matrix elements just as (20) combined with (22) does. The problem now is that it is *not* regularized in the UV sector. To see this, we take a close look at (53) and (54).

In the former matrix element, the second and the third term vanishes because there is an odd number of real and ghost fields and there are no contractions between them. The fourth one also vanishes, because $\boldsymbol{J}^I_{\text{field}}$ is normal ordered. As a result, we are left with the first term and so (53) is the same as unregularized (22b) if we neglect the difference between (33) and (51), which does not provide the UV regularization that we look for.

In the latter matrix element, the second and the third term vanishes because the ghost operators are normal ordered. The fourth term does not vanish, but it is independent of the spin orientation because there are no contractions of ghost fields on states without ghost particles. Thus, it cannot regularize the s_z-dependent final result for field angular momentum of the electron. In fact, by knowing that the ghost fermionic propagator is given by (24) with m_o replaced by Λ [23,24], one can easily check that contribution of the fourth term to the final result vanishes for the very same reason why $\mathcal{F}^{\mu\nu}_{\text{sym}}$ does not contribute to (43). So, after dropping this term, (54) is the same as unregularized (22c) if we note that (24) still holds for Lagrangian density (50).

Therefore, we are left with the option of a formal modification of propagators in the spirit of the Pauli-Villars regularization. Such an approach comes in different flavors. For example, one can modify the electromagnetic propagator through

$$\frac{1}{p^2-\lambda^2+\mathrm{i}0} \to \frac{1}{p^2-\lambda^2+\mathrm{i}0} - \frac{1}{p^2-\Lambda^2+\mathrm{i}0}. \quad (55)$$

Alternatively, one may modify the fermionic propagator through either

$$\frac{\gamma\cdot p+m_o}{p^2-m_o^2+\mathrm{i}0} \to (\gamma\cdot p+m_o)\left(\frac{1}{p^2-m_o^2+\mathrm{i}0} - \frac{1}{p^2-\Lambda^2+\mathrm{i}0}\right) \quad (56)$$

or

$$\frac{\gamma\cdot p+m_o}{p^2-m_o^2+\mathrm{i}0} \to \frac{\gamma\cdot p+m_o}{p^2-m_o^2+\mathrm{i}0} - \frac{\gamma\cdot p+\Lambda}{p^2-\Lambda^2+\mathrm{i}0}, \quad (57)$$

where, e.g., the former option is discussed in [25] while the latter one in [26]. We have checked that those three ways of regularization lead to the same final result. Therefore, we will employ (56) as it yields the simplest analytical expressions. The Pauli-Villars-regularized one-loop part of (43) then reads

$$\langle J^i_{\text{field}}\rangle^{\lambda\Lambda}_{\mathbf{\Omega}s} = -2\mathrm{i}e_o^2 s_z\delta^{i3}\int\frac{d^4p}{(2\pi)^4}\frac{2(p^0-m_o)^2+(p_1)^2+(p_2)^2}{[(p-f)^2-\lambda^2+\mathrm{i}0]^2} \cdot\left(\frac{1}{p^2-m_o^2+\mathrm{i}0} - \frac{1}{p^2-\Lambda^2+\mathrm{i}0}\right). \quad (58)$$

To evaluate it, we join the propagators' denominators through the formula

$$\frac{1}{AB^2} = \int_0^1 da\, db\, \delta(a+b-1) \frac{2b}{(aA+bB)^3}, \tag{59}$$

shift the integration variable to make the resulting denominator p^2-dependent, Lorentz-average the numerator of the integrand with

$$p^\mu p^\nu \to \frac{\eta^{\mu\nu}}{4} p^2, \tag{60}$$

and perform Wick rotation to arrive at

$$\langle J^i_{\text{field}} \rangle^{\lambda\Lambda}_{\boldsymbol{\Omega}s} = -\frac{e_o^2 s_z \delta^{i3}}{8\pi^2} [I(\lambda, m_o) - I(\lambda, \Lambda)], \tag{61a}$$

$$I(\lambda, \chi) = \int_0^1 ds \frac{2s(1-s)^2}{(1-s)[(\chi/m_o)^2 - s] + s(\lambda/m_o)^2}. \tag{61b}$$

Combining this with

$$\lim_{\lambda\to 0} I(\lambda, m_o) = 1, \quad \lim_{\substack{\Lambda\to\infty \\ \lambda\to 0}} I(\lambda, \Lambda) = 0, \tag{62}$$

which can be straightforwardly shown, we finally get

$$\langle J^i_{\text{field}} \rangle_{\boldsymbol{\Omega}s} = \lim_{\substack{\Lambda\to\infty \\ \lambda\to 0}} \langle J^i_{\text{field}} \rangle^{\lambda\Lambda}_{\boldsymbol{\Omega}s} + O(e_o^4) = -s_z \delta^{i3} \frac{e_o^2}{8\pi^2} + O(e_o^4). \tag{63}$$

The same result is obtained if one first integrates (58) over p^0 using the residue theorem and then performs radial and angular integrations in the $\boldsymbol{p}$-space in an arbitrary order.

4. Discussion

We have shown that a finite value for angular momentum stored in electric and magnetic fields of the electron can be obtained in quantum electrodynamics. This is a non-trivial result because individual components of electron's angular momentum need not be finite [8–10]. Interestingly, our calculations of this fundamentally-important not-so-intuitive quantity have not employed any renormalization procedure.

The complication, which we have encountered, is that we have actually obtained two finite one-loop results for field angular momentum of the electron: (49) and (63) in 3D cutoff- and Pauli-Villars-regularized QED. Using $e_o = e + O(e^3)$, they can be written as

$$\langle J^i_{\text{field}} \rangle_{\boldsymbol{\Omega}s} = -s_z \delta^{i3} \frac{\alpha}{3\pi} + O(\alpha^2) \tag{64a}$$

and

$$\langle J^i_{\text{field}} \rangle_{\boldsymbol{\Omega}s} = -s_z \delta^{i3} \frac{\alpha}{2\pi} + O(\alpha^2), \tag{64b}$$

respectively.

We suspect that the disagreement is caused by the lack of recovery of the Lorentz symmetry upon removal of the 3D cutoff regularization. Such a regularization, unlike the Pauli-Villars regularization, explicitly breaks this symmetry in the intermediate steps of the calculations. As a result, we are inclined to think that Pauli-Villars-regularized result (64b) provides the correct value of field angular momentum of the electron. At the same time, we hope that these two findings will stimulate discussion of regularization (in)dependence of QED calculations. We also hope that they will motivate experimental studies of field angular momentum of the electron.

These results can be now compared to the classical estimation that we have discussed in Sec. 1. Such a comparison is of interest if one aims at getting intuitive insights into the QED calculations. We find two curious differences between (5) and (64).

First, (5) overestimates field angular momentum of the electron by roughly three orders of magnitude.

Second, field angular momentum of the electron is anti-aligned with the electron's spin in (64). The opposite is observed in (5). This is illustrated in Fig. 1, where the assumed downward orientation of the magnetic moment $\boldsymbol{\mu}$ implies upward orientation of the spin of a negatively charged particle.

The first difference can be made less severe by increasing the cutoff r_c. For example, one may try

$$r_c = O(r_0) \to O\left(\frac{r_0}{\alpha}\right). \tag{65}$$

This modification makes sense because QED corrections to the Coulomb field are non-negligible at distances smaller than the reduced Compton wavelength, which is given by $1/m = r_0/\alpha$ [18]. In other words, the classically-motivated cutoff used in Sec. 1 leads to employment of expression (2) for the Coulomb field well beyond its range of applicability.

Such a fix, however, has no influence on the second difference. If we now assume that classical expression (5) captures long-distance contribution to field angular momentum of the electron, we could conclude from (64) that the short-distance contribution to this quantity is crucial for getting the right answer. This is the reason why classical estimations, akin to what we have presented in Sec. 1, will always have to be incomplete.

Next, at the risk of stating the obvious, we mention that it would be most desirable to have an experimental measurement of field angular momentum of the electron. Given the fact that we deal here with a gauge invariant observable, whose expectation value is finite, it seems reasonable to assume that such a measurement may be feasible. Perhaps one difficulty associated with it would be that the quantity of interest here is rather small. The same, however, can be said about the Schwinger's correction to the electron's magnetic moment, which was measured about seven decades ago (see e.g. [27]). Therefore, the big open question is how one can experimentally approach such a quantity.

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Appendix A. Conventions

We use the Minkowski metric $\eta = \mathrm{diag}(+ - - -)$ and choose $\varepsilon^{0123} = +1 = \varepsilon^{123}$. Greek and Latin indices take values 0, 1, 2, 3 and 1, 2, 3, respectively, when they refer to the components of 4- and 3-vectors. The Einstein summation convention is applied to those indices. Moreover, 3-vectors are written in bold, e.g. $x = (x^\mu) = (x^0, \boldsymbol{x})$.

We employ Heaviside-Lorentz units and set $\hbar = c = 1$. The fine-structure constant is then given by

$$\alpha = \frac{e^2}{4\pi}. \tag{A.1}$$

We work in the standard representation of γ matrices. The normalization condition of single-electron eigenstates of the Dirac Hamiltonian, say $|\boldsymbol{p}s\rangle$ with $\boldsymbol{p}$ being the electron's 3-momentum and s being its spin state, is $\langle \boldsymbol{p}s|\boldsymbol{p}'s'\rangle = \delta(\boldsymbol{p} - \boldsymbol{p}')\delta_{ss'}$. The u_s bispinors, which appear in contractions on external lines, are normalized such that

$$u_s = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for } s_z = +1/2, \; u_s = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ for } s_z = -1/2. \tag{A.2}$$

They are eigenstates of the z-component of the one-particle fermionic spin angular momentum operator, $\mathrm{i}\varepsilon^{3mn}\gamma^m\gamma^n/4$, to the eigenvalue s_z. Moreover, $(\mathrm{i}\gamma^\mu\partial_\mu - m_0)e^{-\mathrm{i}m_0 t}u_s = 0$.

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